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On Some Univalent Integral Operators

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In this paper some new subclasses of Bazilevic functions are defined and it is shown that these classes are preserved under certain integral operators. © 1987 Academic Press, Inc.

1. INTRODUCTION

Let S, K, S^*, and C denote the classes of analytic functions f: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are respectively univalent, close-to-convex, starlike (with respect to the origin), and convex in the unit disc E. In [2], Janowski introduced the class $P[A, B]$. For A and B, $-1 \leqslant B < A \leqslant 1$, a function p, analytic in E with $p(0) = 1$ belongs to the class $P[A, B]$ if $p(z)$ is subordinate to $(1 + Az)/(1 + Bz)$. Also, given C and D, $-1 \leqslant D < C \leqslant 1$, $C[C, D]$, and $S^*[C, D]$ denote the classes of functions of analytic in E with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ such that $(zf'(z))'/f'(z) \in P[C, D]$ and $zf'(z)/f(z) \in P[C, D]$, respectively. For $C = 1$ and $D = -1$, we note that $C[1, -1] = C$ and $S^*[1, -1] = S^*$. Silvia [6] defined the class $K[A, B; C, D]$ as follows: A function f: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, analytic in E, is said to be in the class $K[A, B; C, D]$, $-1 \leqslant B < A \leqslant 1$: $-1 \leqslant D < C \leqslant 1$, if there exists a $g \in C[C, D]$ such that $f'(z)/g'(z) \in P[A, B]$. It is clear that $K[1, -1; 1, -1] = K$ and $K[A, B; C, D] \subset K \subset S$.

We now define the following:

DEFINITION 1.1. Let $\alpha > 0$ be real and f: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f \in B_\alpha[A, B; C, D]$ if, and only if, there exists a function $g \in S^*[C, D]$ such that $(zf'(z)\, f^{\alpha-1}(z)/g^\alpha(z)) \in P[A, B]$, $z \in E$.

We note that $B_\alpha[1, -1; 1, -1] = B(\alpha)$, a subclass of Bazilevic functions defined in [7]. Also $B_1[A, B; C, D] = K[A, B; C, D]$.

DEFINITION 1.2. Let $\alpha > 0$ be real and f: $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ be analytic in E. Then $f \in B_1[A, B; \alpha]$ if and only if $(zf'(z)\, f^{\alpha-1}(z)/z^\alpha) \in P[A, B]$.

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For $A=1$ and $B=-1$, we obtain a subclass $B_1[1, -1, \alpha]=B_1(\alpha)$ of Bazilevic functions defined in [7]. In this paper, we give some theorems concerning the classes $B_\alpha[A, B; C, D]$, $B[A, B; \alpha]$, and $K[A, B; C, D]$.

2. Preliminary Results

We shall need the following extension of Libera's lemma [3].

Lemma 2.1 [4]. *Let N and D be analytic in E, D maps E onto a many-sheeted starlike region. $N(0)=0=D(0)$, $N'(0)=D'(0)=1$, and*

$$\left\{\frac{N'(z)}{D'(z)}\right\} \in P[A, B].$$

Then

$$\left\{\frac{N(z)}{D(z)}\right\} \in P[A, B].$$

Lemma 2.2. *Let $p_1(z)$ and $p_2(z) \in P[A, B]$. Then, for α, β any positive reals,*

$$\frac{1}{\alpha+\beta}[\alpha p_1(z)+\beta p_2(z)] \in P[A, B].$$

Proof. This follows easily from the definition.

Lemma 2.3. *Let $f \in B_1[A, B, \alpha]$, α a positive integer. Then $(f(z)/z)^\alpha \in P[A, B]$.*

Proof. Since $f \in B_1[A, B, \alpha]$, it follows that $zf'(z)/f^{1-\alpha}(z)z^\alpha \in P[A, B]$. Now

$$\frac{zf'(z)}{f^{1-\alpha}(z)z^\alpha}=\frac{d\{(f(z))^\alpha\}/dz}{d(z^\alpha)/dz}=\frac{N'(z)}{D'(z)}$$

$$\Rightarrow \frac{N(z)}{D(z)}=\frac{(f(z))^\alpha}{z^\alpha} \in P[A, B].$$

Lemma 2.4. *Let α and m be any positive integers and $f \in S^*[A, B]$, $g \in S^*[C, D]$, where $-1 \leqslant B<A \leqslant 1$ and $-1 \leqslant D<C \leqslant 1$. Then the function F defined by*

$$(F(z))^\alpha=\frac{\alpha+m}{(g(z))^m}\int_0^z \zeta^{m-1}(f(\zeta))^\alpha\, d\zeta \tag{2.1}$$

is starlike of order β, *where* $0<\beta<1$ *for* $|z|<\sigma$, *where* σ *is given by*

$$\sigma=\frac{L+\sqrt{L^2-\alpha(1-\beta)K}}{-K}, \qquad K<0, \quad L>0, \tag{2.2}$$

and

$$L=\tfrac{1}{2}\left[(m-\alpha\beta)(D-B)+\alpha(D-A)+m(B-C)\right], \tag{2.3}$$

$$K=-\left[mB(D-C)+\alpha D(A-B)\right]. \tag{2.4}$$

Proof. Let

$$J(z)=\int_0^z \zeta^{m-1}(f(\zeta))^\alpha\, d\zeta.$$

So

$$F(z)=\frac{\alpha+m}{(g(z))^m}J(z)$$

and

$$\alpha\frac{zF'(z)}{F(z)}=\frac{zJ'(z)}{J(z)}-m\frac{zg'(z)}{g(z)}$$

or

$$\frac{zF'(z)}{F(z)}=\frac{m}{\alpha}+\frac{1}{\alpha}\left\{\frac{zJ'(z)-mJ(z)}{J(z)}\right\}-\frac{m}{\alpha}\frac{zg'(z)}{g(z)}. \tag{2.5}$$

Let

$$\frac{N(z)}{D(z)}=\frac{(1/\alpha)\{zJ'(z)-mJ(z)\}}{J(z)}.$$

Then $N(0)=0=D(0)$. By a lemma due to Bernardi [1], $D(z)$ is a $(m+\alpha+1)$-valent starlike. Also,

$$\frac{N'(z)}{D'(z)}=\frac{zf'(z)}{f(z)}\in P[A,B],$$

and this implies that $p_1\equiv N(z)/D(z)\in P[A,B]$, by using Lemma 2.1. Now $p_2\equiv zg'(z)/g(z)\in P[C,D]$, since $g\in S^*[C,D]$; therefore, we have from (2.5),

$$\frac{zF'(z)}{F(z)}=\frac{m}{\alpha}+p_1-\frac{m}{\alpha}p_2. \tag{2.6}$$

But it is known [4] that for $p \in P[A, B]$,

$$\frac{1-Ar}{1-Br} \leqslant \operatorname{Re} p(z) \leqslant \frac{1+Ar}{1+Br}, \qquad |z| \leqslant r < 1. \tag{2.7}$$

Using (2.7), the relation (2.6) yields

$$\operatorname{Re}\left\{\frac{zF'(z)}{F(z)}\right\} \geqslant \frac{m}{\alpha} + \frac{1-Ar}{1-Br} - \frac{m}{\alpha}\frac{1+Cr}{1+Dr}.$$

Now $\operatorname{Re}\{zF'(z)/F(z) - \beta\} \geqslant 0$ whenever

$$\left(\frac{m}{\alpha} - \beta\right) + \frac{1-Ar}{1-Br} - \frac{m}{\alpha}\frac{1+Cr}{1+Dr} \geqslant 0$$

or

$$(m-\alpha\beta)[1-(D-B)r - BDr^2] + \alpha[1+(D-A)r - ADr^2]$$
$$-m[1+(C-B)r - BCr^2] > 0$$

or

$$Kr^2 + 2Lr + \alpha(1-\beta) > 0,$$

where

$$K = -[mB(D-C) + \alpha D(A-B)]$$

and

$$L = \tfrac{1}{2}\,[(m-\alpha\beta)(D-B) + \alpha(D-A) + m(B-C)] > 0.$$

In other words, $\operatorname{Re} zF'(z)/F(z) > \beta$, for $|z| = r < \sigma$, where

$$\sigma = \frac{L + \sqrt{L^2 - \alpha(1-\beta)K}}{-K}, \qquad K < 0,\ L > 0.$$

Special Case. When $g(z) = z$ in Lemma 2.4, we have

$$(F(z))^{\alpha} = \frac{\alpha+m}{z^m}\int_0^z \zeta^{m-1}(f(\zeta))^{\alpha}\, d\zeta,$$

and

$$\frac{zF'(z)}{F(z)} = \left(\frac{1}{\alpha}\{zJ'(z) - mJ(z)\}/J(z)\right) \in P[A, B],$$

where $J(z)$ is as defined in Lemma 2.4. Hence $F \in S^*[A, B]$.

LEMMA 2.5. *Let $g \in S^*[C, D]$, $h \in C[C, D]$, and let H be defined by*

$$H(z) = \int_0^z \left(\frac{g(\zeta)}{\zeta}\right)^\alpha \left(\frac{h(\zeta)}{\zeta}\right)^\beta d\zeta, \qquad \alpha + \beta = 1; 0 < \alpha, \beta \leqslant 1,$$

Then $H \in C[C, D]$.

Proof.

$$zH'(z) = z\left(\frac{g(z)}{z}\right)^\alpha \left(\frac{h(z)}{z}\right)^\beta$$
$$= (g(z))^\alpha \cdot (h(z))^\beta.$$

So

$$1 + z\frac{H''(z)}{H'(z)} = \frac{(zH'(z))'}{H'(z)} = \alpha\frac{zg'(z)}{g(z)} + \beta\frac{zh'(z)}{h(z)}$$
$$= \alpha p_1(z) + \beta p_2(z), \qquad p_1, p_2 \in P[C, D],$$

where we have used the fact that $g \in S^*[C, D]$ and $h \in C[C, D] \subset S^*[C, D]$. Hence, from Lemma 2.2, we conclude that $H \in C[C, D]$.

3. MAIN RESULTS

We now proceed to prove

THEOREM 3.1. *Let m and α be any positive integers and $f \in B_\alpha[A, B; C, D]$ and let F be defined as*

$$(F(z))^\alpha = \frac{(m+\alpha)}{z^m} \int_0^z \zeta^{m-1} (f(\zeta))^\alpha \, d\zeta. \tag{3.1}$$

Then $F \in B_\alpha[A, B; C, D]$.

Proof. Differentiating (3.1), we have

$$\alpha\frac{zF'(z)}{F^{1-\alpha}(z)} = \frac{(m+\alpha)}{z^m}\left\{z^m f^\alpha(z) - m\int_0^z \zeta^{m-1} f^\alpha(z)\, d\zeta\right\}. \tag{3.2}$$

Let $g \in S^*[C, D]$ such that $zf'(z)/f^{1-\alpha}(z)\, g^\alpha(z) \in P[A, B]$ and let

$$(G(z))^\alpha = \frac{(m+\alpha)}{z^m} \int_0^z \zeta^{m-1} (g(\zeta))^\alpha \, d\zeta. \tag{3.3}$$

Then, from the special case of Lemma 2.4, $G \in S^*[C, D]$.

From (3.2) and (3.3), we can write

$$\frac{zF'(z)}{F^{1-\alpha}(z)\,G^{\alpha}(z)}=\frac{(1/\alpha)\{z^m f^{\alpha}(z)-m\int_0^z \zeta^{m-1} f^{\alpha}(\zeta)\,d\zeta\}}{\int_0^z g^{\alpha}(\zeta)\,d\zeta}=\frac{N(z)}{D(z)},$$

say. Now,

$$\frac{N'(z)}{D'(z)}=\frac{zf'(z)}{f^{1-\alpha}(z)\,g^{\alpha}(z)}\in P[A, B],$$

and so, using Lemma 2.1, we see that

$$\frac{N(z)}{D(z)}=\frac{zF'(z)}{F^{1-\alpha}(z)\,G^{\alpha}(z)}\in P[A, B].$$

Hence $F\in B_{\alpha}[A, B; C, D]$.

Special Cases. (i) For $\alpha=1$, $m=1$, we see that $f\in K[A, B; C, D]$ and F, defined by Libera's integral operator, also belongs to $K[A, B; C, D]$.

(ii) For $m=1$, we have $F\in B_{\alpha}[A, B; C, D]$, where

$$(F(z))^{\alpha}=\frac{\alpha+1}{z}\int_0^z (f(\zeta))^{\alpha}\,d\zeta, \qquad f\in B_{\alpha}[A, B; C, D].$$

We can also prove

THEOREM 3.2. *Let m and α be any positive integers and let $f\in B_1[A, B, \alpha]$. Let F be defined by* (3.1). *Then $F\in B_1[A, B, \alpha]$.*

THEOREM 3.3. *Let $f\in B_1[A, B, \alpha]$, $\alpha>0$. Then the function F_1 defined by $(F_1(z))^{\alpha+\beta}=z^{\beta}(f(z))^{\alpha}$ belongs to $B_1[A, B; \alpha+\beta]$, $\beta\geqslant 0$.*

Proof. From the definition of F_1, we have

$$\frac{(\alpha+\beta)\,zF_1'(z)}{(F_1(z))^{1-(\alpha+\beta)}}=\beta z^{\beta}(f(z))^{\alpha}+\alpha z^{\beta}(zf'(z))\cdot f^{\alpha-1}(z),$$

and therefore,

$$\begin{aligned}\frac{zF_1'(z)}{(F_1(z))^{1-(\alpha+\beta)}\cdot z^{\alpha+\beta}}&=\frac{1}{(\alpha+\beta)}\left\{\beta\left(\frac{f(z)}{z}\right)^{\alpha}+\alpha\frac{zf'(z)}{(f(z))^{1-\alpha}\cdot z^{\alpha}}\right\}\\&=\frac{1}{(\alpha+\beta)}\{\beta p_1(z)+\alpha p_2(z)\}, \qquad p_1, p_2\in P[A, B].\\&=p(z)\in P[A, B), \qquad \text{from} \quad \text{Lemma 2.2.}\end{aligned}$$

Hence $F_1\in B_1[A, B; \alpha+\beta]$.

THEOREM 3.4. *Let $f \in K[A, B; C, D]$ with respect to $h \in C[C, D]$. Let $g \in S^*[C, D]$ and for $\alpha + \beta = 1$, $0 \leqslant \alpha, \beta \leqslant 1$, let F be defined as*

$$F(z) = \int_0^z \left(\frac{f(\zeta)}{\zeta}\right)^\alpha \left(\frac{g(\zeta)}{\zeta}\right)^\beta d\zeta.$$

Then $F \in K[A, B; C, D]$ with respect to H defined by

$$H(z) = \int_0^z \left(\frac{h(\zeta)}{\zeta}\right)^\alpha \left(\frac{g(\zeta)}{\zeta}\right)^\beta d\zeta.$$

Proof. Fom Lemma 2.5, $H \in C[C, D]$. Now

$$\frac{F'(z)}{H'(z)} = \left(\frac{f(z)}{z}\right)^\alpha \left(\frac{g(z)}{z}\right)^\beta \Big/ \left(\frac{h(z)}{z}\right)^\alpha \left(\frac{g(z)}{z}\right)^\beta$$

$$= \left(\frac{f(z)}{h(z)}\right)^\alpha, \qquad 0 < \alpha \leqslant 1.$$

Since $f \in K[A, B; C, D]$ with respect to $H \in C[C, D]$, so $f'(z)/h'(z) \in P[A, B] \Rightarrow f(z)/h(z) \in P[A, B]$. Hence $F'(z)/H'(z) \in P[A, B]$ and $F \in K[A, B; C, D]$.

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